Quantum Violation: Beyond Clauser-Horne-Shimony-Holt Inequality

Hoshang Heydari*
Institute of Quantum Science, Nihon University,
1-8 Kanda-Surugadai, Chiyoda-ku, Tokyo 101-8308, Japan
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The best upper bound for the violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality was first derived by Tsirelson. For increasing number of ± 1 valued observables on both sites of the correlation experiment, Tsirelson obtained the Grothendieck's constant ($\mathcal{K}_G \approx 1.73 \pm 0.06$) as a limit for the maximal violation. In this paper, we construct a generalization of the CHSH inequality with four ± 1 valued observables on both sites of a correlation experiment and show that the quantum violation approaching 1.58. Moreover, we estimate the maximal quantum violation of a correlation experiment for large and equal number of ± 1 valued observables on both sites. In this case, the maximal quantum violation converges to $\sqrt{3} \approx 1.73$ for very large n, which coincides with the approximate value of Grothendieck's constant.

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I. INTRODUCTION

The seminal paper of Einstein, Podolsky, and Rosens (EPR)[1], and Schrödinger's article [2] on quantum correlations of entangled states as well as Bell [3] subsequent discovery that quantum theory is incompatible with any locally realistic, hidden variable theory have generated substantial discussions and many experiments on the nature of quantum non-locality. The violation of Bell's inequality was the first mathematically sharp criterion for entanglement. A quantum state is said to be unentangled, separable if and only if it can be written as a convex combination of product states. In some cases, however, this criterion fails to detect any entanglement. The standard example of the Bell inequality is the CHSH inequality [4], which refers to correlation experiments with two ± 1 valued observables on two sites. In this paper, we will only discuss the CHSH type inequality. However, there is an infinite hierarchy of such Bell type inequalities, which can basically be classified by specifying the type of correlation experiments they deal with. The CHSH inequalities are by far the best-studied cases of Bell inequalities. The essential assumption leading to any Bell inequality is the existence of a local realistic model, which describes the outcomes of a certain class of correlation measurements. Various aspects of the hierarchy of Bell inequalities have already been investigated. Garg and Mermin[5], for instance, have resumed the idea of Bell and discussed systems with maximal correlation. Gisin [6] investigated setups with more than two dichotomic observables per site with arbitrary states, which we will also discuss in the following section. N-particle generalizations of the CHSH inequality were first proposed by Mermin[7], and further developed by Ardehali[8], Belinskii and Klyshko[9], and others[10, 11]. The best upper bound for the violation of the CHSH inequality, first derived by Tsirelson[12], is obtained by squaring the Bell operator and utilizing the variance inequality[14]. In the case of more than two dichotomic observables per site only very little is known about the limit. In particular there is yet no explicit characterization of the extremal inequalities, although constructing some inequalities, e.g. by chaining CHSH inequalities[15] is not difficult. However, Tsirelson[12, 13] recognized that the quantum correlation functions, which are in general rather cumbersome objects, can be reexpressed in terms of finite dimensional vectors in Euclidean space. For two observables on one site and an arbitrary number on the other, Tsirelson showed that the maximal quantum violation is $\sqrt{2}$. However, for an increasing number of observables on both sites, he obtained the upper bound for Grothendieck's constant $\mathcal{K}_G$ (≈ 1.78), known from the geometry of Banach spaces, as the limit for the maximal violation[16]. In particular, $\mathcal{K}_G$ is the smallest number, such that, for all integers $n \geq 2$, all $n \times n$ real matrices $[a_{ij}]$, and all $s_1, \cdots, s_n, t_1, \cdots, t_n \in \mathbf{R}$ such that $|s_i|, |t_j| \leq 1$ for which $|\sum_{i,j} a_{ij} s_i t_j| \leq 1$, it is true that $\left|\sum_{i,j} a_{ij} \langle x_i, y_j \rangle\right| \leq \mathcal{K}_G$, where $x_1, \cdots, x_n$, $y_1, \cdots, y_n$ such that $\|x_i\|, \|y_j\| \leq 1$ are vectors in a real Hilbert space. Tsirelson [12] showed that comparisons between probabilities in classical physics and probabilities in quantum mechanics yield discrepancy measures $\mathcal{K}_n$ for finite $n \times n$ real matrices that approach Grothendieck's constant K_G for very lager n. The exact value of $\mathcal{K}_G$ is unknown. A lower bound of $\pi/2$ was established by Grothendieck [17]. In a recent paper, P. C. Fishburn and J. A. Reeds [18] showed that $\mathcal{K}_{q(q-1)} \geq (3q-1)/(2q-1)$ for $q \geq 2$ and

*Electronic address: hoshang@imit.kth.se

$\mathcal{K}_{20} \geq \frac{10}{7}$; $n = 20$ is the smallest known n for which $\mathcal{K}_n > \sqrt{2}$. In this paper, we will construct a generalization of the CHSH inequality with four observables on both sites and show that maximal quantum violation approaches $\sqrt{\frac{5}{2}} \simeq 1.58$. Moreover, we will estimate the maximal quantum violation for very large numbers of observables per site in a correlation experiment.

II. THE STRUCTURE OF THE SET OF QUANTUM CORRELATIONS

In this section, we will define CHSH inequality and Tsirelson inequality as the best upper bound for the violation of the CHSH inequality. First, let us define CHSH inequality as follows. Let $\mathrm{Cor}_{\mathcal{C}}(n,m)$ denote the set of classically representable matrices, whose matrix elements are

$$\langle \mathrm{X}_k, \mathrm{Y}_l \rangle_c = \int \mathrm{X}_k(\varphi)\mathrm{Y}_l(\varphi)d(\varphi), \tag{1}$$

where $\mathrm{X}_k, \mathrm{Y}_l$ are random variables satisfying $|\mathrm{X}_k| \leq 1, |\mathrm{Y}_l| \leq 1$. Then, the CHSH inequality is defined by

$$|\langle \mathrm{X}_1, \mathrm{Y}_1 \rangle_c + \langle \mathrm{X}_1, \mathrm{Y}_2 \rangle_c + \langle \mathrm{X}_2, \mathrm{Y}_1 \rangle_c - \langle \mathrm{X}_2, \mathrm{Y}_2 \rangle_c| \leq 2. \tag{2}$$

The CHSH inequality holds for any local-realistic theory. However, quantum correlation violate the CHSH inequality, that is, let us consider the following observables

$$\widehat{\mathrm{X}}_k = \widehat{\mathrm{X}}_k^{(1)} \otimes \mathrm{I}^{(2)} \text{ and } \widehat{\mathrm{Y}}_l = \mathrm{I}^{(1)} \otimes \widehat{\mathrm{Y}}_l^{(2)} \tag{3}$$

for all $k = 1, 2, ..., n$ and $l = 1, 2, ..., m$, where $\mathrm{I}^{(j)}$ is identity operator on the Hilbert space $\mathcal{H}_j$, such that the following relations are satisfied by these operators $[\widehat{\mathrm{X}}_k, \widehat{\mathrm{Y}}_l] = 0$ for all k and l that is $\widehat{\mathrm{X}}_k$ is compatible with each $\widehat{\mathrm{Y}}_l$. Hence for an arbitrary state $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2$, the quantum correlation is measurable and $\|\widehat{\mathrm{X}}_j\| \leq 1, \|\widehat{\mathrm{Y}}_k\| \leq 1$ for all k and l. Thus we can define the quantum correlation matrix $\mathcal{C}$ as

$$\mathcal{C} = (\langle \widehat{\mathrm{X}}_k, \widehat{\mathrm{Y}}_l \rangle_\rho)_{k=1,2,...,n;\ l=1,2,...,m}, \tag{4}$$

where $\langle \widehat{\mathrm{X}}_k, \widehat{\mathrm{Y}}_l \rangle_\rho = \mathrm{Tr}(\rho \widehat{\mathrm{X}}_k \widehat{\mathrm{Y}}_l)$. Now, let the convex set $\mathrm{Cor}_{\mathcal{Q}}(n,m)$ be quantum -representable matrices of some quantum observables $\widehat{\mathrm{X}}_k, \widehat{\mathrm{Y}}_l$ as describe above. The geometrical description of this convex set follows from the following theorem [20, 22]: The matrix $\mathcal{C}$ belongs to the set $\mathrm{Cor}_{\mathcal{Q}}(n,m)$ if and only if there exist vectors $\mathrm{a}_1, \mathrm{a}_2, ..., \mathrm{a}_n$ and $\mathrm{b}_1, \mathrm{b}_2, ..., \mathrm{b}_m$in Euclidean space of dimension $\min(n,m)$, such that $\|\mathrm{a}_k\| \leq 1$, $\|\mathrm{b}_l\| \leq 1$ and $\mathrm{a}_k \cdot \mathrm{b}_l = \langle \widehat{\mathrm{X}}_k, \widehat{\mathrm{Y}}_l \rangle_\rho$, for all k and l.

Now, let us define for $n = m = 2$ the Bell operator with the same structure as the combination which appears on the CHSH inequality as $\mathcal{B}_{2,2} = \widehat{\mathrm{X}}_1\widehat{\mathrm{Y}}_1 + \widehat{\mathrm{X}}_1\widehat{\mathrm{Y}}_2 + \widehat{\mathrm{X}}_2\widehat{\mathrm{Y}}_1 - \widehat{\mathrm{X}}_2\widehat{\mathrm{Y}}_2$. Then, we have

$$\mathcal{B}_{2,2}^2 = 4\mathrm{I} - [\widehat{\mathrm{X}}_1, \widehat{\mathrm{X}}_2][\widehat{\mathrm{Y}}_1, \widehat{\mathrm{Y}}_2], \tag{5}$$

where we have assumed $\widehat{\mathrm{X}}_k^2 = \widehat{\mathrm{Y}}_l^2 = \mathrm{I}$ for all k and l. From this inequality we can get the CHSH inequality as

$$\begin{aligned} \mathrm{Tr}(\rho\mathcal{B}_{2,2}) &= \langle \widehat{\mathrm{X}}_1, \widehat{\mathrm{Y}}_1 \rangle_\rho + \langle \widehat{\mathrm{X}}_1, \widehat{\mathrm{Y}}_2 \rangle_\rho \\ &\quad + \langle \widehat{\mathrm{X}}_2, \widehat{\mathrm{Y}}_1 \rangle_\rho - \langle \widehat{\mathrm{X}}_2, \widehat{\mathrm{Y}}_2 \rangle_\rho \leq 2, \end{aligned} \tag{6}$$

whenever $[\widehat{\mathrm{X}}_1, \widehat{\mathrm{X}}_2] = [\widehat{\mathrm{Y}}_1, \widehat{\mathrm{Y}}_2] = 0$. The upper bound which gives the maximal violation of CHSH is called Tsirelson inequality and is given by

$$\mathrm{Tr}(\rho\mathcal{B}_{2,2}) \leq 2\sqrt{2}, \tag{7}$$

where for any observable $\widehat{\mathrm{Z}} = \widehat{\mathrm{X}}, \widehat{\mathrm{Y}}$ and $\|\widehat{\mathrm{Z}}_k\| \leq 1, \|\widehat{\mathrm{Z}}_l\| \leq 1$ for all $k, l = 1, 2$, we have estimate $|[\widehat{\mathrm{Z}}_1, \widehat{\mathrm{Z}}_2]\|$ as follows

$$\begin{aligned} |[\widehat{\mathrm{Z}}_1, \widehat{\mathrm{Z}}_2]\| &\leq \|\widehat{\mathrm{Z}}_1\widehat{\mathrm{Z}}_2\| + \|\widehat{\mathrm{Z}}_2\widehat{\mathrm{Z}}_1\| \\ &\leq \|\widehat{\mathrm{Z}}_1\|\|\widehat{\mathrm{Z}}_2\| + \|\widehat{\mathrm{Z}}_2\|\|\widehat{\mathrm{Z}}_1\| \leq 2. \end{aligned} \tag{8}$$

We will use this estimation in the next section when we derive an inequality for the case $n = m = 4$ and will try to estimate the maximal violation of generalized CHSH inequality.

III. MAXIMAL QUANTUM VIOLATION

The maximal violation for an increasing number of observables on both sites of a correlation experiment is still an unsolve problem. However, Tsirelson has obtained the Grothendieck's constant as a limit for the maximal violation. In this case we have

$$\mathrm{Cor}_{\mathcal{C}}(n,m) \subset \mathrm{Cor}_{\mathcal{Q}}(n,m). \tag{9}$$

For example, the CHSH inequality provides a hyperplane separating the polyhedron $\mathrm{Cor}_{\mathcal{C}}(2,2)$ from the quantum realizable matrix $\mathcal{R}_{2,2} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, such that $\mathcal{R}_{2,2} \in \mathrm{Cor}_{\mathcal{Q}}(2,2)$. So it is natural to ask how much $\mathrm{Cor}_{\mathcal{Q}}(n,m)$ exceeds $\mathrm{Cor}_{\mathcal{C}}(n,m)$. Let $\mathcal{K}(n,m)$ be the smallest number having this property, that is

$$\mathrm{Cor}_{\mathcal{Q}}(n,m) \subset \mathcal{K}(n,m)\mathrm{Cor}_{\mathcal{C}}(n,m). \tag{10}$$

Then this sequence increases with n and m. It was found by Tsirelson, from geometrical description of the set $\mathrm{Cor}_{\mathcal{Q}}(n,m)$, that

$$\mathcal{K} = \lim_{n,m\to\infty} \mathcal{K}(n,m) \tag{11}$$

coincides with the Grothendieck's constant $\mathcal{K}_G \leq \frac{\pi}{2\ln(1+\sqrt{2})} \approx 1.78$ known from the geometry of Banach spaces. In the next section, we will construct an generalized CHSH inequality with more than two observables per site and show that for an arbitrary state this inequality has an upper bound which is lager than the upper bound for CHSH inequality and it approaches the approximative Grothendieck's constant.

IV. BEYOND TSIRELSON INEQUALITY

In the case of correlation experiments with more than two dichotomic observables per site only very little is known. So we will here go beyond this limit by allowing four dichotomic observables per site. In this case ($n = m = 4$), we will consider an inequality that provides a hyperplane separating the polyhedron $\mathrm{Cor}_{\mathcal{C}}(4,4)$ from the quantum realizable matrix

$$\mathcal{R}_{4,4} = \mathcal{R}_{2,2} \otimes \mathcal{R}_{2,2} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \tag{12}$$

such that $\mathcal{R}_{4,4} \in \mathrm{Cor}_{\mathcal{Q}}(4,4)$. Now, let $X_k = \pm 1$ and $Y_k = \pm 1$ for all indices $k = 1,2,3,4$. Then we get the following inequality

$$\begin{aligned} &\mathrm{X}_1(\mathrm{Y}_1 + \mathrm{Y}_2 + \mathrm{Y}_3 + \mathrm{Y}_4) + \mathrm{X}_2(\mathrm{Y}_1 - \mathrm{Y}_2 + \mathrm{Y}_3 - \mathrm{Y}_4) \\ &+\mathrm{X}_3(\mathrm{Y}_1 + \mathrm{Y}_2 - \mathrm{Y}_3 - \mathrm{Y}_4) + \mathrm{X}_4(\mathrm{Y}_1 - \mathrm{Y}_2 - \mathrm{Y}_3 + \mathrm{Y}_4) \leq 8. \end{aligned}$$

Based on this inequality, we obtain the generalized CHSH inequality as in the equation (2)

$$\begin{aligned} &|\langle \mathrm{X}_1, \mathrm{Y}_1\rangle_c + \langle \mathrm{X}_1, \mathrm{Y}_2\rangle_c + \langle \mathrm{X}_1, \mathrm{Y}_3\rangle_c + \langle \mathrm{X}_1, \mathrm{Y}_4\rangle_c \\ &+\langle \mathrm{X}_2, \mathrm{Y}_1\rangle_c - \langle \mathrm{X}_2, \mathrm{Y}_2\rangle_c + \langle \mathrm{X}_2, \mathrm{Y}_3\rangle_c - \langle \mathrm{X}_2, \mathrm{Y}_4\rangle_c \\ &+\langle \mathrm{X}_3, \mathrm{Y}_1\rangle_c + \langle \mathrm{X}_3, \mathrm{Y}_2\rangle_c - \langle \mathrm{X}_3, \mathrm{Y}_3\rangle_c - \langle \mathrm{X}_3, \mathrm{Y}_4\rangle_c \\ &+\langle \mathrm{X}_4, \mathrm{Y}_1\rangle_c - \langle \mathrm{X}_4, \mathrm{Y}_2\rangle_c - \langle \mathrm{X}_4, \mathrm{Y}_3\rangle_c + \langle \mathrm{X}_4, \mathrm{Y}_4\rangle_c| \leq 8. \end{aligned} \tag{13}$$

Then we can get the following Bell operator with the same structure as the combination which appears on the CHSH inequality as

$$\begin{aligned} \mathcal{B}_{4,4} &= \begin{pmatrix} \widehat{\mathrm{X}}_1 & \widehat{\mathrm{X}}_1 & \widehat{\mathrm{X}}_3 & \widehat{\mathrm{X}}_4 \end{pmatrix} \mathcal{R}_{4,4} \begin{pmatrix} \widehat{\mathrm{Y}}_1 \\ \widehat{\mathrm{Y}}_2 \\ \widehat{\mathrm{Y}}_3 \\ \widehat{\mathrm{Y}}_4 \end{pmatrix} \\ &= \widehat{\mathrm{X}}_1(\widehat{\mathrm{Y}}_1 + \widehat{\mathrm{Y}}_2 + \widehat{\mathrm{Y}}_3 + \widehat{\mathrm{Y}}_4) + \widehat{\mathrm{X}}_2(\widehat{\mathrm{Y}}_1 - \widehat{\mathrm{Y}}_2 + \widehat{\mathrm{Y}}_3 - \widehat{\mathrm{Y}}_4) \\ &+ \widehat{\mathrm{X}}_3(\widehat{\mathrm{Y}}_1 + \widehat{\mathrm{Y}}_2 - \widehat{\mathrm{Y}}_3 - \widehat{\mathrm{Y}}_4) + \widehat{\mathrm{X}}_4(\widehat{\mathrm{Y}}_1 - \widehat{\mathrm{Y}}_2 - \widehat{\mathrm{Y}}_3 + \widehat{\mathrm{Y}}_4). \end{aligned} \tag{14}$$

Now, we will apply the same procedure as in the case of finding the upper bound for the violation of the CHSH inequality, by squaring the Bell operator $\mathcal{B}_{4,4}^2$ which is given by

$$\begin{aligned} = \; & [\widehat{\mathrm{X}}_1,\widehat{\mathrm{X}}_2]([\widehat{\mathrm{Y}}_2,\widehat{\mathrm{Y}}_3]+[\widehat{\mathrm{Y}}_4,\widehat{\mathrm{Y}}_3]+[\widehat{\mathrm{Y}}_4,\widehat{\mathrm{Y}}_1]+[\widehat{\mathrm{Y}}_2,\widehat{\mathrm{Y}}_1]) \\ & +[\widehat{\mathrm{X}}_1,\widehat{\mathrm{X}}_3]([\widehat{\mathrm{Y}}_4,\widehat{\mathrm{Y}}_2]+[\widehat{\mathrm{Y}}_4,\widehat{\mathrm{Y}}_1]+[\widehat{\mathrm{Y}}_3,\widehat{\mathrm{Y}}_1]+[\widehat{\mathrm{Y}}_3,\widehat{\mathrm{Y}}_2]) \\ & +[\widehat{\mathrm{X}}_1,\widehat{\mathrm{X}}_4]([\widehat{\mathrm{Y}}_3,\widehat{\mathrm{Y}}_4]+[\widehat{\mathrm{Y}}_2,\widehat{\mathrm{Y}}_1]+[\widehat{\mathrm{Y}}_3,\widehat{\mathrm{Y}}_1]+[\widehat{\mathrm{Y}}_2,\widehat{\mathrm{Y}}_4]) \\ & +[\widehat{\mathrm{X}}_2,\widehat{\mathrm{X}}_3]([\widehat{\mathrm{Y}}_2,\widehat{\mathrm{Y}}_4]+[\widehat{\mathrm{Y}}_4,\widehat{\mathrm{Y}}_3]+[\widehat{\mathrm{Y}}_3,\widehat{\mathrm{Y}}_1]+[\widehat{\mathrm{Y}}_1,\widehat{\mathrm{Y}}_2]) \\ & +[\widehat{\mathrm{X}}_2,\widehat{\mathrm{X}}_4]([\widehat{\mathrm{Y}}_2,\widehat{\mathrm{Y}}_3]+[\widehat{\mathrm{Y}}_3,\widehat{\mathrm{Y}}_1]+[\widehat{\mathrm{Y}}_1,\widehat{\mathrm{Y}}_4]+[\widehat{\mathrm{Y}}_4,\widehat{\mathrm{Y}}_2]) \\ & +[\widehat{\mathrm{X}}_3,\widehat{\mathrm{X}}_4]([\widehat{\mathrm{Y}}_1,\widehat{\mathrm{Y}}_4]+[\widehat{\mathrm{Y}}_2,\widehat{\mathrm{Y}}_1]+[\widehat{\mathrm{Y}}_4,\widehat{\mathrm{Y}}_3]+[\widehat{\mathrm{Y}}_3,\widehat{\mathrm{Y}}_2]) \\ & +\{\widehat{\mathrm{X}}_1,\widehat{\mathrm{X}}_2\}(\{\widehat{\mathrm{Y}}_1,\widehat{\mathrm{Y}}_3\}-\{\widehat{\mathrm{Y}}_2,\widehat{\mathrm{Y}}_4\}) \\ & +\{\widehat{\mathrm{X}}_1,\widehat{\mathrm{X}}_3\}(\{\widehat{\mathrm{Y}}_1,\widehat{\mathrm{Y}}_2\}-\{\widehat{\mathrm{Y}}_3,\widehat{\mathrm{Y}}_4\}) \\ & +\{\widehat{\mathrm{X}}_1,\widehat{\mathrm{X}}_4\}(\{\widehat{\mathrm{Y}}_1,\widehat{\mathrm{Y}}_4\}-\{\widehat{\mathrm{Y}}_2,\widehat{\mathrm{Y}}_3\}) \\ & +\{\widehat{\mathrm{X}}_2,\widehat{\mathrm{X}}_3\}(\{\widehat{\mathrm{Y}}_2,\widehat{\mathrm{Y}}_3\}-\{\widehat{\mathrm{Y}}_1,\widehat{\mathrm{Y}}_4\}) \\ & +\{\widehat{\mathrm{X}}_2,\widehat{\mathrm{X}}_4\}(\{\widehat{\mathrm{Y}}_3,\widehat{\mathrm{Y}}_4\}-\{\widehat{\mathrm{Y}}_1,\widehat{\mathrm{Y}}_2\}) \\ & +\{\widehat{\mathrm{X}}_3,\widehat{\mathrm{X}}_4\}(\{\widehat{\mathrm{Y}}_2,\widehat{\mathrm{Y}}_4\}-\{\widehat{\mathrm{Y}}_1,\widehat{\mathrm{Y}}_3\})+16\mathrm{I}. \end{aligned} \tag{15}$$

In similarity with the CHSH inequality we can chose $[\widehat{\mathrm{X}}_k,\widehat{\mathrm{X}}_l]=[\widehat{\mathrm{Y}}_k,\widehat{\mathrm{Y}}_l]=0$ for all k and l, that is, these are commuting observables on both sites. The result is the following inequality

$$\begin{aligned} \mathcal{B}_{4,4}^2 \; = \; & 16\mathrm{I}+\{\widehat{\mathrm{X}}_1,\widehat{\mathrm{X}}_2\}(\{\widehat{\mathrm{Y}}_1,\widehat{\mathrm{Y}}_3\}-\{\widehat{\mathrm{Y}}_2,\widehat{\mathrm{Y}}_4\}) \\ & +\{\widehat{\mathrm{X}}_1,\widehat{\mathrm{X}}_3\}(\{\widehat{\mathrm{Y}}_1,\widehat{\mathrm{Y}}_2\}-\{\widehat{\mathrm{Y}}_3,\widehat{\mathrm{Y}}_4\}) \\ & +\{\widehat{\mathrm{X}}_1,\widehat{\mathrm{X}}_4\}(\{\widehat{\mathrm{Y}}_1,\widehat{\mathrm{Y}}_4\}-\{\widehat{\mathrm{Y}}_2,\widehat{\mathrm{Y}}_3\}) \\ & +\{\widehat{\mathrm{X}}_2,\widehat{\mathrm{X}}_3\}(\{\widehat{\mathrm{Y}}_2,\widehat{\mathrm{Y}}_3\}-\{\widehat{\mathrm{Y}}_1,\widehat{\mathrm{Y}}_4\}) \\ & +\{\widehat{\mathrm{X}}_2,\widehat{\mathrm{X}}_4\}(\{\widehat{\mathrm{Y}}_3,\widehat{\mathrm{Y}}_4\}-\{\widehat{\mathrm{Y}}_1,\widehat{\mathrm{Y}}_2\}) \\ & +\{\widehat{\mathrm{X}}_3,\widehat{\mathrm{X}}_4\}(\{\widehat{\mathrm{Y}}_2,\widehat{\mathrm{Y}}_4\}-\{\widehat{\mathrm{Y}}_1,\widehat{\mathrm{Y}}_3\}). \end{aligned} \tag{16}$$

An estimation of this inequality gives

$$\mathrm{Tr}(\rho\mathcal{B}_{4,4}) \leq 8, \tag{17}$$

where the observables satisfies $\|\widehat{\mathrm{X}}_k\| \leq 1$ and $\|\widehat{\mathrm{Y}}_l\| \leq 1$ for all $k,l=1,2,3,4$. Moreover, we have supposed that the anticommutators does not vanish for these observables. Note that $\widehat{\mathrm{X}}_k\widehat{\mathrm{X}}_l=\pm\widehat{\mathrm{X}}_l\widehat{\mathrm{X}}_k$ and $\widehat{\mathrm{Y}}_k\widehat{\mathrm{Y}}_l=\pm\widehat{\mathrm{Y}}_l\widehat{\mathrm{Y}}_k$ implies $\widehat{\mathrm{Y}}_k\widehat{\mathrm{Y}}_l=\widehat{\mathrm{X}}_k\widehat{\mathrm{X}}_l=0$. If we keep this in mind, then we can get an upper bound of the maximal quantum violation for the equation (13). If we estimate the inequality without letting any of the observables commute on both sites, then we get

$$\mathrm{Tr}(\rho\mathcal{B}_{4,4}) \leq \sqrt{160}=4\sqrt{10}. \tag{18}$$

Now, we would like to compare this result with Tsirelson upper bound for CHSH inequality, where $\frac{1}{2}\mathrm{Tr}(\rho\mathcal{B}_{2,2}) \leq \sqrt{2} \simeq 1.41$. For the generalized CHSH inequality with four observables on both sites we get

$$\frac{1}{8}\mathrm{Tr}(\rho\mathcal{B}_{4,4}) \leq \sqrt{\frac{5}{2}} \simeq 1.58, \tag{19}$$

where we have used $\|\{\widehat{\mathrm{Z}}_k,\widehat{\mathrm{Z}}_l\}\| \leq 2 \;\; \|[\widehat{\mathrm{Z}}_k,\widehat{\mathrm{Z}}_l]\| \leq 2$ for $\widehat{\mathrm{Z}}=\widehat{\mathrm{X}},\widehat{\mathrm{Y}}$ and for all $k,l=1,2,3,4$. This estimation differ from Tsirelson upper bound for the CHSH inequality because of existence of commutators and anti-commutators in the square of the Bell operator (15). However, this is what we expect to get from Tsirelson idea that quantum correlation should approach the Grothendieck's constant as the number of observables increase on both sites of a correlation experiment. Moreover, it is very difficult to show that these upper bound is tight, that is, the equality is approached for some quantum state, this needs further investigations.

We can also generalize this result in a straightforward manner into a generalized CHSH inequality with $n=m=2^d$ dichotomic observables per site. In this case, we will consider an inequality that provides a hyperplane separating the

polyhedron $\mathrm{Cor}_{\mathcal{C}}(2^d, 2^d)$ from the quantum realizable matrix

$$\mathcal{R}_{2^d,2^d} = \overbrace{\mathcal{R}_{2,2} \otimes \cdots \otimes \mathcal{R}_{2,2}}^{d}, \tag{20}$$

such that $\mathcal{R}_{2^d,2^d} \in \mathrm{Cor}_{\mathcal{Q}}(2^d, 2^d)$. Based on this idea, we can get the following Bell operator

$$\mathcal{B}_{2^d,2^d} \;=\; \begin{pmatrix} \widehat{\mathrm{X}}_1 & \widehat{\mathrm{X}}_1 & \cdots & \widehat{\mathrm{X}}_{2^d} \end{pmatrix} \mathcal{R}_{2^d,2^d} \begin{pmatrix} \widehat{\mathrm{Y}}_1 \\ \widehat{\mathrm{Y}}_2 \\ \vdots \\ \widehat{\mathrm{Y}}_{2^d} \end{pmatrix}. \tag{21}$$

Now, we will apply the same procedure as in the case of four observables per site by squaring the Bell operator $\mathcal{B}_{2^d,2^d}$. Then we can write $\mathcal{B}^2_{2^d,2^d}$ in terms of commutator and anticommutator. However, note that this estimation is only valid for $d \geq 2$ since for $d = 1$ we do not have any anticommutator in our expression for the Bell operator. Next, we chose $[\widehat{\mathrm{X}}_k, \widehat{\mathrm{X}}_l] = [\widehat{\mathrm{Y}}_k, \widehat{\mathrm{Y}}_l] = 0$ for all k and l. An estimation of this inequality gives

$$\mathrm{Tr}(\rho\mathcal{B}_{2^d,2^d}) \;\leq\; (4\frac{2^d(2^d-1)}{2}2^{d-1} + 2^{2d})^{\frac{1}{2}} = 2^{\frac{3}{2}d}, \tag{22}$$

where the first term is a contribution from the anticommutators and second term from the identity operators, which are the squares of the observables, that is $\widehat{\mathrm{X}}^2_k = \widehat{\mathrm{Y}}^2_l = \mathrm{I}$ for all $k, l = 1, 2, \ldots, 2^d$. Now, we can get an upper bound on the generalized CHSH inequality (21) if we estimate the inequality without letting any of the observables commute on both sites, that is

$$\begin{aligned}\mathrm{Tr}(\rho\mathcal{B}_{2^d,2^d}) \;&\leq\; (4\frac{2^d(2^d-1)}{2}2^d + 4\frac{2^d(2^d-1)}{2}2^{d-1} + 2^{2d})^{\frac{1}{2}} \\ &=\; 2^d(3\cdot 2^d - 2)^{\frac{1}{2}},\end{aligned} \tag{23}$$

where the first term is a contribution from the commutators. Thus in the general case with 2^d observables per site we get

$$\frac{1}{2^{\frac{3}{2}d}}\mathrm{Tr}(\rho\mathcal{B}_{2^d,2^d}) \leq 2^{-\frac{d}{2}}(3\cdot 2^d - 2)^{\frac{1}{2}}. \tag{24}$$

Let us analysis this inequality. For CHSH inequality with two observables per site this estimation does not work since there is no contribution from anticommutator in this inequality. In the case of four observables per site, we get the same result as in equation (19) that is $\frac{1}{2^3}\mathrm{Tr}(\rho\mathcal{B}_{2^2,2^2}) \leq \sqrt{\frac{5}{2}}$. And finally, for a very large number of observables per site, that is whenever $d \to \infty$, we have $\lim_{d\to\infty} \frac{1}{2^{\frac{3d}{2}}}\mathrm{Tr}(\rho\mathcal{B}_{2^d,2^d}) \leq \sqrt{3} \approx 1.73$. This is less than upper bound for the Grothendieck's constant (≈ 1.782). However, it almost coincides with the approximate value of Grothendieck's constant. Moreover, it can be seen that in our inequality the maximal quantum violation increases with the number of observables per site and approaches the maximum value $\sqrt{3}$.

V. CONCLUSION

In this paper, we have constructed an especial type of the CHSH inequality with four observables per site of a correlation experiment and we have shown that for arbitrary state the quantum violation is higher than the Tsirelson bound for CHSH inequality. Moreover, we have estimated the maximal quantum violation for very large but equal number observables on both sites of a correlation experiment. The estimation shows that in this case the maximal quantum violation converges to $\sqrt{3} \approx 1.73$, which coincides with Grothendieck's constant. This result also can be seen as an indirect estimation of Grothendieck's constant. However, this estimation needs further investigation. The approximative value of this constant was pointed out by Tsirelson [22]. In this paper, he also has discussed the difficultly to find a quantum state that gives the maximal quantum violation for a given CHSH type inequality. We should mention that the CHSH inequality does not include any anticommutator but a generalized CHSH inequality does include both commutators and anticommutators. In our estimation, we have assumed that the values of these commutators and anticommutators do coexist simultaneously and contribute to an estimation of the maximal quantum violation. We also should mention that exact value of the Grothendieck is not known yet and our results could be interesting for the research on this subject.

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